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Black-hole solution without curvature singularity

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Abstract

An exact solution of the vacuum Einstein field equations over a nonsimply-connected manifold is presented. This solution is spherically symmetric and has no curvature singularity. It can be considered to be a regularization of the Schwarzschild solution over a simply-connected manifold, which has a curvature singularity at the center. Spherically symmetric collapse of matter in $\mathbb{R}^4$ may result in this nonsingular black-hole solution, if quantum-gravity effects allow for topology change near the center.

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I. INTRODUCTION

The vacuum Einstein field equations over $\mathcal{M}_4 = \mathbb{R}^4$ have a spherically symmetric solution, the Schwarzschild black-hole solution [1] with its maximal extension [2–4]. Recently, a related solution has been found for the vacuum Einstein field equations over $\widetilde{\mathcal{M}}_4$, a particular nonsimply-connected manifold [5]. The goal of the present article is to discuss the connection between these two vacuum solutions. The comparison of these two solutions will, in fact, suggest a black-hole-type solution, where the curvature singularity is eliminated by a spacetime defect, i.e., a "hole" in spacetime.

II. SINGULAR BLACK-HOLE SOLUTION

The extended Schwarzschild–Kruskal–Szekeres solution [1–3] has a metric given by the following line element ($G_N = c = 1$):

$$ds^2 \Big|_{\text{SKS}}^{(r>0)} = 32\, M^3\, \frac{\exp\big[-r/(2M)\big]}{r} \Big(- dv^2 + du^2 \Big) + r^2 \Big(d\theta^2 + \sin^2\theta\; d\phi^2 \Big)\,, \tag{1a}$$

with $\theta \in [0,\, \pi]$, $\phi \in [0,\, 2\pi)$, and r implicitly given in terms of the coordinates $u \in \mathbb{R}$ and $v \in \mathbb{R}$ by the relation

$$\left(\frac{r}{2M} - 1\right)\, \exp\big[r/(2M)\big]\, \Big|_{\text{SKS}} = u^2 - v^2 > -1\,, \tag{1b}$$

for $r > 0$. The explicit expression for r from (1b) is

$$r\, \Big|_{\text{SKS}} = 2M\left(1 + W_0\left[\frac{u^2 - v^2}{e}\right]\right)\,, \tag{1c}$$

in terms of the principal branch of the Lambert W–function, $W_0[z]$, which gives the principal solution for w in $z = w\,\exp[w]$. Recall that $W_0[x]$ is real for $x \geq -1/e$.

The solution (1) has $M > 0$ as a free parameter and the corresponding topology is

$$\mathcal{M}_{\text{SKS}} = \mathbb{R}^2 \times S^2\,. \tag{2}$$

The solution approaches a physical singularity for $r \to 0$, as shown by the divergence of the Kretschmann scalar (defined in terms of the Riemann curvature tensor),

$$K\, \Big|_{\text{SKS}} \equiv\; R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma}\, \Big|_{\text{SKS}} = 48\, \frac{M^2}{r^6}\,. \tag{3}$$

Further details can be found in, e.g., Ref. [4].

III. NONSINGULAR BLACK-HOLE SOLUTION

Consider, now, the vacuum Einstein field equations over

$$\widetilde{\mathcal{M}}_4 \;=\; \mathbb{R} \times \widetilde{\mathcal{M}}_3\,, \tag{4a}$$

where the 3-space $\widetilde{\mathcal{M}}_3$ is a noncompact, orientable, nonsimply-connected manifold without boundary. Specifically, there are the following homeomorphisms:

$$\widetilde{\mathcal{M}}_3 \simeq \mathbb{R}P^3 - \{\text{point}\} \simeq SO(3) - \{\text{point}\}\,, \tag{4b}$$

with $\mathbb{R}P^3$ the 3-dimensional real projective plane and $SO(3)$ the Lie group of unit-determinant orthogonal 3×3 matrices. Here, and in the following, $\widetilde{\mathcal{M}}$ with tilde indicates a nonsimply-connected manifold [having a nontrivial first homotopy group, $\pi_1(\widetilde{\mathcal{M}}) \neq 0$], whereas $\mathcal{M}$ without tilde stands for a simply-connected manifold [having $\pi_1(\mathcal{M}) = 0$].

The explicit construction of $\widetilde{\mathcal{M}}_3$ has been given in Ref. [5]: start from $\mathbb{R}^3$ with the standard Cartesian coordinates and Euclidean metric and, then, perform surgery on $\mathbb{R}^3$ by removing the interior of a ball B_b with radius b and identifying antipodal points on the boundary of the ball, $\partial B_b = S_b^2$ (2-sphere with radius b). In Ref. [5], also appropriate coordinate charts are reviewed. There is, for example, one chart (labeled $n = 2$) with 'radial' coordinate $Y_2 \in (-\infty, +\infty)$ instead of the standard radial coordinate $r \in [b, \infty)$ and angular coordinates (X_2, Z_2) having restricted ranges. For the moment, the subscript '2' on these chart-2 coordinates will be omitted.

The new exact vacuum solution over $\widetilde{\mathcal{M}}_4$ involves two parameters, b and M, which, for the present purpose, are taken to be related as follows:

$$0 < b < 2M\,. \tag{5}$$

The defect solution for $2M < b$ has been given by Eqs. (4.1), (5.1b), and (5.1c) in Ref. [5] for chart-2 coordinates and inspection shows it to be precisely of the form of the standard Schwarzschild solution in the exterior region ($\widetilde{r} > b > 2M$) if $Y^2 + b^2$ and $\widetilde{r}^2$ are identified. This observation agrees with Birkhoff's theorem [4] and, in turn, allows us to use some of the techniques of the Kruskal–Szekeres procedure in our search for a solution with parameters (5).

It turns out that the Kruskal–Szekeres procedure is useful to eliminate the apparent singularities at the Schwarzschild horizon (having $\zeta = 2M$, with a notation to be explained shortly) but is not obviously well suited to deal with the spacetime defect (having $\zeta = b$). For this reason, we will present our black-hole-type solution without curvature singularity in terms of two sets of coordinates, one set of coordinates appropriate to the spacetime defect and another set of coordinates further out.

Start, then, with an *Ansatz* in terms of Kruskal–Szekeres-type coordinates, valid for $\zeta > b$ and having a metric given by the following line element:

$$ds^2\,\Big|^{(\zeta > b)} = 32\, M^3\, \frac{\exp\big[-\zeta/(2M)\big]}{\zeta}\,\Big(-dV_\pm^2 + dU_\pm^2\Big) + \zeta^2\,\Big(dZ^2 + \sin^2 Z\; dX^2\Big)\,, \tag{6a}$$

with ζ implicitly given in terms of the coordinates $U_\pm \in \mathbb{R}$ and $V_\pm \in \mathbb{R}$ by the relation

$$\left(\frac{\zeta}{2M} - 1\right)\, \exp\big[\zeta/(2M)\big] = \big(U_\pm^2 - V_\pm^2\big) > \left(\frac{b}{2M} - 1\right)\, \exp\big[b/(2M)\big]\,, \tag{6b}$$

for $\zeta > b > 0$. Again, ζ can be obtained explicitly by use of the Lambert function $W_0[x]$. The previous chart-2 coordinates $Y,\, T \in (-\infty,\, +\infty)$ from Eqs. (2.6) and (2.8) in Ref. [5] are recovered as follows:

$$Y \;=\; \begin{cases} +\sqrt{\zeta\big(V_+,\, U_+\big)^2 - b^2}\,, \\ -\sqrt{\zeta\big(V_-,\, U_-\big)^2 - b^2}\,, \end{cases} \tag{6c}$$

$$\frac{T}{4M} \;=\; \begin{cases} \tanh^{-1}\big[V_\pm/U_\pm\big] \;\; \text{for} \;\; |U_\pm| > |V_\pm|\,, \\ \tanh^{-1}\big[U_\pm/V_\pm\big] \;\; \text{for} \;\; |U_\pm| \leq |V_\pm|\,. \end{cases} \tag{6d}$$

Relations (6c) and (6d) can be readily inverted for appropriate spacetime regions, in particular regions I and II from Eq. (31.17) in Ref. [4].

From the *Ansatz* (6a), written in terms of the chart-2 coordinates $\{Y,\, T\} \in (-\infty,\, +\infty)$ by using the bottom-row expression on the right-hand side of (6d), it is possible to obtain the interior metric with the defect at $\zeta \equiv \sqrt{b^2 + Y^2} = b$ included,

$$\begin{aligned} ds^2\Big|^{(0\,\leq\, Y^2\,<\,4M^2-b^2)}_{\text{chart-2}} \;&=\; +\left(\frac{2M}{\sqrt{b^2+Y^2}} - 1\right)\, dT^2 - \left(\frac{2M}{\sqrt{b^2+Y^2}} - 1\right)^{-1}\, \frac{Y^2}{b^2+Y^2}\;\, dY^2 \\ &\quad +(b^2+Y^2)\left(dZ^2 + \sin^2 Z\; dX^2\right). \end{aligned} \tag{7a}$$

Similarly, one obtains from the same *Ansatz* (6a), but now using the top-row expression on the right-hand side of (6d), the exterior metric,

$$\begin{aligned} ds^2\Big|^{(Y^2\,>\,4M^2-b^2)}_{\text{chart-2}} \;&=\; -\left(1 - \frac{2M}{\sqrt{b^2+Y^2}}\right)\, dT^2 + \left(1 - \frac{2M}{\sqrt{b^2+Y^2}}\right)^{-1}\, \frac{Y^2}{b^2+Y^2}\, dY^2 \\ &\quad +(b^2+Y^2)\left(dZ^2 + \sin^2 Z\; dX^2\right). \end{aligned} \tag{7b}$$

For the boundary between the exterior and interior regions [$\zeta = 2M$ between regions I and II, as mentioned below (6d)], the *Ansatz* (6a) in terms of $(V_\pm\,, U_\pm)$ coordinates can simply be kept as it is,

$$\begin{aligned} ds^2\Big|^{(2M-\Delta L\,<\,\zeta\,<\,2M+\Delta L)}_{\text{chart-2}} \;&=\; 32\, M^3\; \frac{\exp\big[-\zeta/(2M)\big]}{\zeta}\, \Big(-dV_\pm^2 + dU_\pm^2\Big) \\ &\quad +\zeta^2\left(dZ^2 + \sin^2 Z\; dX^2\right), \end{aligned} \tag{7c}$$

where ζ is now given by (6b) and ΔL is a small enough positive length, so that $b < 2M - \Delta L$. The coordinate transformations for the overlap regions between (7c) and (7a) or (7b) are given by (6c) and (6d).

Note that, strictly speaking, (7b) could be omitted if the range of (7c) is extended to $\zeta \in (b,\, \infty)$, with the case $\zeta = b$ covered by (7a). Perhaps it is even possible to use only (7c), but now for the range $\zeta \in [b,\, \infty)$ and with a point-wise identification of the two sheets at $\zeta = b$ given by $\big(V_+,\, U_+\big) \cong \big(V_-,\, U_-\big)$. This last possibility will be explored in a follow-up paper, but, here, we continue with (7) as it stands.

For the interior metric (7a), T has become spacelike and Y timelike, similar to what happens for the standard Schwarzschild solution (cf. Fig. 31.1 in Ref. [4]), but now Y ranges from $-\infty$ to $+\infty$, unlike the usual radial coordinate r. Still, this timelike coordinate Y is part of a topologically nontrivial manifold (see below), which may have important implications as will be discussed in Sec. IV.

The Riemann curvature tensor $R^{\kappa}{}_{\lambda\mu\nu}(T,\,X,\,Y,\,Z)$ from (7a) is found to be even in Y and finite at $Y=0$; see the explicit expressions given by Eq. (5.5) in Ref. [5]. The Ricci tensor $R_{\mu\nu}(T,\,X,\,Y,\,Z)$ from (7) vanishes identically (the same holds for the Ricci scalar R) and, hence, the vacuum Einstein equations are solved.

The results of the previous paragraph show that the vanishing YY–component of the metric (7a) at $Y=0$ is a coordinate artifact, as confirmed by the transformation to the Schwarzschild-type coordinate $\widetilde{r}=\sqrt{b^2+Y^2}$, which gives an invertible metric at $\widetilde{r}=b$. [Careful: the usual radial coordinate would be $r=b+|Y|$. In other words, the metric from the right-hand side of (7a) for $M=0$ is *not* equivalent to the standard Minkowski metric for the usual spherical coordinates, where the ball B_b has been removed by surgery as explained in the second paragraph of this section. This last metric has, in fact, a Ricci scalar R proportional to $r^{-1}\,\delta(r-b)$, whereas the metric (7a) has $R=0$ throughout.]

With the Riemann tensor of the vacuum solution (7), the Kretschmann scalar is found to be given by

$$K \;\equiv\; R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma} = 48\,\frac{M^2}{\zeta^6}\,, \tag{8}$$

with $\zeta^2 \equiv b^2+Y^2$ for the interior metric (7a) or the exterior metric (7b) and ζ given by (6b) for the boundary metric (7c). For all three metrics from (7), the Kretschmann scalar remains finite because $b>0$.

As discussed in Sec. 2 of Ref. [5], the proper description of $\widetilde{\mathcal{M}}_3$ requires three coordinate charts. In the present context, there are, then, the coordinates $(V_{n,\pm},\,U_{n,\pm},\,X_n,\,Z_n)$ and $(T_n,\,X_n,\,Y_n,\,Z_n)$, for $n=1,2,3$, with $(V_{2,\pm},\,U_{2,\pm},\,X_2,\,Z_2)=(V_\pm,\,U_\pm,\,X,\,Z)$ as used in (7c) and $(T_2,\,X_2,\,Y_2,\,Z_2)\equiv(T,\,X,\,Y,\,Z)$ as used in (7a) and (7b). Hence, the interior spacetime manifold from (7a) extended to all charts and the exterior spacetime manifold from (7b) extended to all charts have the same topology, namely, $\widetilde{\mathcal{M}}_4$ from (4a) in terms of $\widetilde{\mathcal{M}}_3$ from (4b).

IV. DISCUSSION

The solution (7) of the vacuum Einstein field equations for the topology (4) is the main result of this article. The metrics (7b) and (7c) are, of course, not new [they correspond to the standard Schwarzschild–Kruskal–Szekeres metric], but the interior metric (7a) for appropriate coordinates is new, as it describes the spacetime defect at $\zeta=\sqrt{b^2+Y^2}=b$, which effectively eliminates the curvature singularity at $\zeta=0$.

The geodesics near $Y=0$ from the metric (7a), as well as other topological matters, will be considered elsewhere. Some qualitative comments will suffice here. For the spacetime

region inside the Schwarzschild event horizon, $Y^2 < 4M^2 - b^2$, the timelike coordinate Y from the metric (7a) is part of the manifold $\widetilde{\mathcal{M}}_3$ as given by (4b) and closed time-like curves (CTCs) exist. These CTCs imply all possible horrors, but, classically, these horrors remain confined within the Schwarzschild horizon. Whether or not CTCs in the interior region are physically acceptable depends on the behavior of the matter fields. But, for now, let us simply return to the vacuum solution (7).

Purely mathematically, the nonsingular vacuum solution (7) with parameter $b > 0$ can be considered to be a "regularization" of part of the singular vacuum solution (1), with all surprises this may entail. It is, however, also possible that this nonsingular solution appears in a physical context.

Start from a nearly flat spacetime (metric approximately equal to the Minkowski metric), where a large amount of matter with total mass M is arranged to collapse in a spherically symmetric way. Within the realm of classical Einstein gravity, one expects to end up with part of the singular solution (1); see, for example, Fig. 32.1.b of Ref. [4]. But, very close to the final curvature singularity, something else may happen due to quantum mechanics.

Considering a precursor mass $\Delta M \sim \hbar/(b\,c) \ll M$ and using typical curvature values from (3) and (8), the local spacetime integral of the action density related to (1) differs from that related to (7) by an amount $\lesssim \hbar$. Then, as argued by Wheeler (cf. Secs. 34.6, 43.4, and 44.3 of Ref. [4] and references therein), the local topology of the manifold may change by a quantum jump if b is sufficiently close to $L_{\text{Planck}} \equiv (\hbar\, G_N/c^3)^{1/2}$, resulting in a transition from a simply-connected manifold to a nonsimply-connected manifold. If the transition amplitude between the different topologies is indeed nonzero for appropriate matter content, quantum mechanics can operate a change between the classical solution (1) and the classical solution (7), thereby removing the curvature singularity.

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